Locally Maximal Embeddings of Graphs
in Orientable Surfaces

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Introduction

We study *2-cell embeddings* $G \hookrightarrow S$ of a fixed graph G in compact orientable surfaces.

Graphs are connected, parallel edges and loop are allowed.

Duke's Interpolation Theorem

Theorem (Duke, 1966)

If a graph G has a 2-cell embedding in orientable surfaces of genera g and h with $g \leq h$, then it has a 2-cell embedding in the surface of genus k for each k such that $g \leq k \leq h$.

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This talk:
Embeddings close to maximum genus and their properties.

Maximum genus

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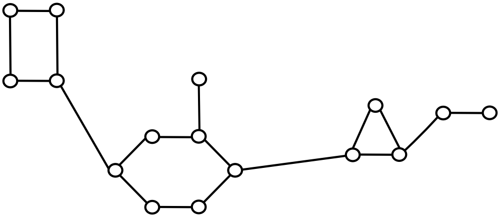

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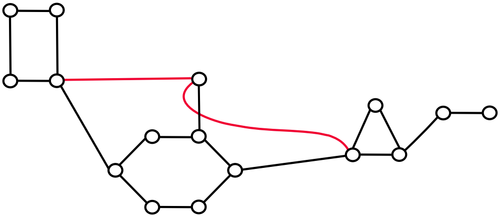

‘Good’ characterisation of maximum genus

By Euler-Poincaré:

$$\gamma_M(G) \leq \beta(G)/2$$

where $\beta(G)$ is the *Betti number* of G (i.e., $\#$ edges $-$ $\#$ vertices $+$ 1).

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Definition

The *deficiency* of G is the quantity $\xi(G) = \beta(G) - 2\gamma_M(G)$.

$$\xi(G) = \min\,(\#\text{faces in a 2-cell embedding of } G) - 1$$

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Definition

Graphs for which $\xi \leq 1$ are called *upper embeddable*.

Equivalently, G is upper embeddable if $\gamma_M = \lfloor \beta/2 \rfloor$.

‘Good’ characterisation of maximum genus

Theorem 1 (Xuong, 1979)

$\xi(G) = \min \#$ of edge-odd components in a cotree of G

Theorem 2 (Nebeský, 1981)

$$\xi(G) = \max\{c(G-A) + oc(G-A) - |A| - 1;\ A \subseteq E(G)\}$$

$c = \#$ of components $\qquad$ $oc = \#$ of components with odd Betti number

Computing the maximum genus

There exist two different polynomial-time algorithms for determining the maximum genus of a graph:

- Glukhov (1981)
  algorithm based on the min-max characterisation of γ_M
- Furst, Gross, McGeoch (1988)
  algorithm based on determining the max # of disjoint adjacent pairs of edges via reduction to the matroid parity problem

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Question

Can we say something more about embeddings close to γ_M?

Representation of graph embeddings by rotations

Every 2-cell embedding $G \hookrightarrow S$ can be represented by a pair of permutations (R, L) acting on the set $D(G)$ of all directed edges, the *darts* of G:

- R ... cyclically permutes darts with the same initial vertex

  $R = \prod_v R_v$ ... the *rotation* of the embedding
- L ... reverses the direction of each dart, i. e., $x \mapsto x^{-1}$

Conversely, given such a pair of permutations

cycles of R ... vertices
cycles of L ... edges
cycles of RL ... faces

incidence ... nonempty intersection

Stratified systems and locally maximal embeddings

Gross & Tucker (1979):
There is a natural notion of adjacency between embeddings of a fixed graph G which gives rise to a stratified system of embeddings.

- Two embeddings of G are adjacent if they can be obtained from each other by moving a single dart to a different position in the local rotation at some vertex.
- Strata are formed by embeddings of the same genus.

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Definition

An embedding $G \hookrightarrow S$ is said to be *locally maximal* if it is not adjacent to any embedding in a higher stratum.

Basic questions about locally maximal embeddings

Questions:

- How deep below γ_M can locally maximal embeddings occur?
- How are they distributed?
- How to construct locally maximal?
- Can we perhaps characterise locally maximal embeddings combinatorially?
- ... and so on.

Rotation moves

Let $G \hookrightarrow S$ be an embedding with rotation R.

- **1. Move of a dart (elementary move):**

$$R_v = (axbAcd) \rightarrow (abAcxd) = R'_v$$

- **2. Interchange of two darts at a vertex:**

$$R_v = (axbAcyd) \rightarrow (aybAcxd) = R'_v$$

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Observation

Every rotation move changes the number of faces of an embedding by -2, 0, *or* $+2$, *and the genus by* -1, 0, *or* 1.

Basic properties: 1. Topological description

Theorem

The following statements are equivalent for a 2-cell embedding $\Pi\colon G \hookrightarrow S$.

(i) Π *is locally maximal, i. e., moving any dart to a different position in the local rotation at some vertex will not increase the genus of* Π.

(ii) *The genus of* Π *does not increase by any rotation move.*

(iii) *Every vertex is incident with at most two faces of* Π.

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Proof.

(iii) $\Rightarrow$ (i):
An elementary move changes # of faces by 0 or ± 2, but -2 is impossible.

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(i) $\Rightarrow$ (iii):

- Let v be a vertex incident with three distinct faces. There is an edge f at v separating two faces. Let $e = R^{-1}(f)$, $g = R(f)$. There is a corner xy at v belonging to a third face.

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- $R_v = (efgAxy)$
  $F_1 = (e^{-1}fB)$, $F_2 = (f^{-1}gC)$, $F_3 = (x^{-1}yD)$

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- $R_v = (efgAxy)$
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- Perform $(efgAxy) \rightarrow (egAxfy)$.

  Then $(e^{-1}fB)(f^{-1}gC)(x^{-1}yD) \rightarrow (x^{-1}fBe^{-1}gCf^{-1}yD)$

□

Basic properties: 2. Absence of strict maxima

Theorem (Gross, Rieper, 1991; Kotrbčík & S., 2014+)

Every 2-cell embedding of a connected graph in an orientable surface can be transformed into a maximum genus embedding by a sequence of dart moves that never decrease the genus.

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Proof.

- Move only *bifacial* edges, i.e., edges on the boundary of two faces.
- Let $\Pi_0\colon G \hookrightarrow S_g$ be an arbitrary embedding of G.
- Take a Xuong spanning tree $T \subseteq G$ corresponding to a maximum genus embedding $\Pi_1\colon G \hookrightarrow S_h$.
- There exists a set X_1 of cotree edges that are bifacial in Π_1 s.t. $G - X_1$ embeds with a single face in S_h.
- X_1 is modified and extended to get a suitable set X_0 s.t. $G - X_0$ embeds with a single face in S_g. Etc.

□

Basic properties: 3. Bounds

Theorem

Let $\lambda(G)$ be the maximum number of faces in a locally maximal embedding of a graph G. Then

$$\lambda(G) \leq \mu(G) + 1$$

where $\mu(G)$ denotes the maximum number of disjoint cycles in G.

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Proof.

Consider a locally maximal embedding Π with max # of faces.

- For a face F, consider its *reduced boundary* $\overline{\mathrm{Bd}}(F)$ consisting of all bifacial edges of $\mathrm{Bd}(F)$. A *seed* of Π is a connected component of $\overline{\mathrm{Bd}}(F)$, where F is any face of Π.

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- Each seed contains a cycle.
- Distinct seeds of Π are disjoint.

Proof (cont.).

- Let H be a seed of Π arising from a face F. Each edge $e \in H$ belongs to the boundary of some other face $F_e \neq F$.

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- If $e, g \in H$ are incident with a common vertex v, then $F_e = F_g$.
- By connectivity, each seed lies within the boundaries of exactly two distinct faces.
- Form a graph $\mathcal{F}_\Pi$ whose vertices are the faces and edges correspond to seeds.
- Estimate the number of vertices of $\mathcal{F}_\Pi$:

$$\lambda(G) \ = \ (\#\text{vertices } \mathcal{F}_\Pi) \ \leq \ (\#\text{edges } \mathcal{F}_\Pi) + 1 \ \leq \ \mu(G) + 1.$$

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Theorem

The following inequalities hold for every connected graph G:

(i) $\gamma(G) \leq \beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G) \leq \beta(G)/2$

(ii) $\beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G) \leq \beta'(G)$

(iii) $\gamma_M(G)/2 \leq \gamma_L(G) \leq \gamma_M(G)$

Lower embeddable graphs

Definition

A graph G is *lower embeddable* if $\gamma_L(G) = \lceil \beta'(G)/2 \rceil$.

A lower embeddable graph has a locally maximal embedding with $\mu(G) + 1$ or $\mu(G)$ faces depending on whether $\beta'(G)$ is even or odd.

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A lower embeddable graph has a locally maximal embedding with $\mu(G) + 1$ or $\mu(G)$ faces depending on whether $\beta'(G)$ is even or odd.

Theorem

Graphs in each of the following classes are lower embeddable:

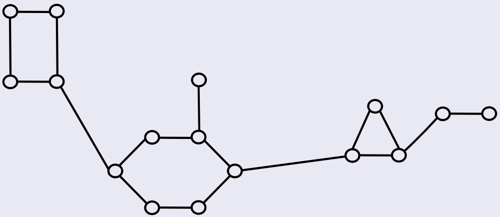
- *complete graphs* K_n
- *complete bipartite graphs* $K_{m,n}$
- *n-cubes* Q_n
- *complete tripartite graphs* $K_{n,n,n}$

Lower embeddable graphs

Proof.

It is sufficient to construct locally maximal embeddings of these graphs with either $\mu + 1$ or μ faces, depending on the parity of β'.

- Construct an inclusion minimal connected spanning subgraph $H \subseteq G$ with either μ or $\mu - 1$ disjoint cycles such that the rest of G can be decomposed into pairs of adjacent edges.
- Embed H in the 2-sphere.
- Add a pair of adjacent edges to increase the genus by 1, and repeat.



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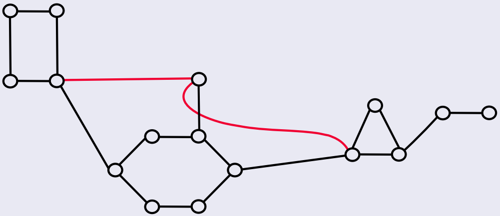

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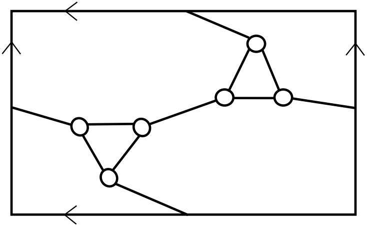

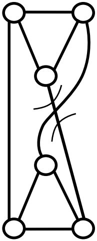

Graphs with no proper locally maximal embeddings

Theorem

The following statements are equivalent for a connected graph G.

(i) $\lceil \beta'(G)/2 \rceil = \lfloor \beta(G)/2 \rfloor$

(ii) *G is both upper embeddable and lower embeddable, and $\gamma_L(G) = \gamma_M(G)$.*

(iii) *Either $\mu(G) \leq 1$, or $\mu(G) = 2$ and $\beta(G)$ is odd.*

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Graphs with $\mu = 1$ were completely described by Lovász (1965).
They include

- bouquets of circles, dipoles, triangles with multiple edges, wheels multiple edges on the outer rim
- complete bipartite graphs $K_{3,n}$ with arbitrarily many edges added within the 3-element partite set, ...

Characterisation of graphs with $\mu = 2$ is not known.

Further results (sample)

Theorem

Locally maximal genus is additive over bridges.

Theorem

If G is a vertex-amalgation of G_1 and G_2, then

$$\gamma_L(G) = \gamma_L(G_1) + \gamma_L(G_2) - c,$$

where the constant c is either 0 *or* 1, *depending on G_1 and G_2.*

Theorem

Let G be a connected graph and e not a bridge of G. Then

$$\gamma_L(G) - 1 \leq \gamma_L(G - e) \leq \gamma_L(G).$$

Future research and problems

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- The natural upper bound on the number of faces in a locally maximal embedding is $\mu(G) + 1$.
- Determining $\mu(G)$ is NP-complete.
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4. Are all 4-edge-connected graphs and all edge-transitive graphs lower-embeddable?

Thank you!